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# A Note on Particular Integral of Differential Equations

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**Abstract**

Linear differential equations with constant coefficients has been one of the modules to Graduate Engineering students pursuing B. Tech Programs from various educational institutions in India under the Course title, Engineering Mathematics. The academicians usually teach particular integral, to obtain solution of non-homogenous differential equations, and at some places, academicians opt for recent publications in order to extend knowledge of the current times keeping the teaching of these methods as their respective perogative. At the same time, students do learn many parallel methods in order to equip themselves as guided through lecture hall discussions and from prescribed text materials. The challenge happens to be evaluation of particular integrals to obtain the complete solution of the non-homogeneous equations that usually encounter vagaries of different methods. Choosing the exact technique to achieve the solution is an endless academic pursuit but students would score different marks with their answers. This paper proposes the choice of the exact method to calculate a particular integral to specific examples and consequences on execution of the proposed methods.

***Keywords:***

Particular integral;
Solution of differential equation;
Challenges in particular integral.

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## 1. Introduction

Let us consider a nth order linear non-homogeneous differential equation with constant coefficients be in the form as $(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n)y = \phi(x)$ (1)

Let (1) can be written as $f(D)y = \phi(x)$ where $f(D) \equiv a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n$(2)

The complete solution of (1) is in the form of $y = C.F + P.I$ where $C.F$ and $P.I$ are known as complementary function and particular integral respectively. The complementary function can be

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obtained by solving the associated homogeneous part of the given differential equation and a particular integral is obtained by solving $\frac{1}{f(D)}\phi(x)$.

A particular integral can be calculated using method of undetermined coefficients, variation of parameters, general integration method and method of operators. The proposed methods may easily applied if the function $\phi(x)$ is an exponential function, trigonometric function or polynomials of degree $k$, while method of operators may be confused to evaluate the particular integral if $\phi(x)$ is the combination of polynomials and trigonometric functions, such as $x^k sin(ax) or x^k \cos(ax)$.

Let us consider $f(D)y = \phi(x)$ be a given differential equation and its $P.I$ can be calculated using the following method of operators.

Theorem-1: If $\phi(x) = e^{ax}$ then $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ provided $f(a) \neq 0$. (3)

If $f(a) = 0$ then apply theorem-4.

Theorem-2: If $\phi(x) = e^{ax}U(x)$ then $\frac{1}{f(D)}[e^{ax}U(x)] = e^{ax}\frac{1}{f(D+a)}U(x)$ (4)

Theorem-3: If $\phi(x) = \begin{Bmatrix} sin\beta x \\ cos\beta x \end{Bmatrix}$ then $\frac{1}{f(D)}\begin{Bmatrix} sin\beta x \\ cos\beta x \end{Bmatrix} = \frac{1}{f_1(D^2)+Df_2(D^2)}\begin{Bmatrix} sin\beta x \\ cos\beta x \end{Bmatrix}$

$$= \frac{1}{f_1(-\beta^2)+f_2(-\beta^2)D}\begin{Bmatrix} sin\beta x \\ cos\beta x \end{Bmatrix} \quad (5)$$

provided $f_1(-\beta^2) + \beta^2 f_2(-\beta^2) \neq 0$

Theorem-4: If $f(a) = 0, f'(a) = 0, f''(a) = 0, \dots, f^{(p-1)}(a) = 0, f^{(p)}(a) \neq 0$ i.e., $a$ is a $p$-fold root of the characteristic equation $f(\lambda) = 0$, then $\frac{1}{f(D)}e^{ax} = \frac{1}{f^{(p)}(a)}x^p e^{ax}$. (6)

If $f_1(-\beta^2) + \beta^2 f_2(-\beta^2) = 0$ in theorem-3 then replace $sin\beta x$ and $cos\beta x$are related to its $e^{i\beta x}$ and hence apply the theorem-4.

Proofs of the above theorems are available in ([5], [6])

Technique for evaluating$P.I$ if $\phi(x)$is a polynomial of degree n

Let $P(x) \equiv p_o + p_1 x + p_2 x^2 + \cdots + p_n x^n$ then using (2)$\frac{1}{f(D)}P(x)$ can be written as

$$\frac{1}{(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n)}P(x)$$

$$= \frac{1}{a_n}\left(1 + \frac{a_{n-1}}{a_n}D + \cdots + \frac{a_1}{a_n}D^{n-1} + \frac{a_0}{a_n}D^n\right)^{-1} P(x)$$

$$= \frac{1}{a_n}(1 + b_1 D + b_2 D^2 + \cdots + b_n D^n + \cdots)P(x) \quad (7)$$

Theorem-5: If $\phi(x) = xV(x)$ then $\frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V$

The proof theorem-5 is mentioned in [4] as follows:

Let $Z = xU(x)$ where $U$ is any function of $x$.

$D(xU) = xDU + U; D^2(xU) = xD^2U + 2DU; D^3(xU) = xD^3U + 3D^2U$

By mathematical induction $D^r(xU) = xD^rU + rD^{r-1}U = xD^rU + \left(\frac{d}{dD}D^r\right)U$

Substituting these values in $f(D)$, we get

$$f(D)xU = xf(D)U + \left\{\frac{d}{dD}f(D)\right\}U \quad (8)$$

Put $f(D)U = V(x)$

Then $U = \frac{1}{f(D)}V$ (9)

Substituting $U$ from (9) in (8)

$$f(D)\left[x\frac{1}{f(D)}V\right] = xf(D)\frac{1}{f(D)}V + \left\{\frac{d}{dD}f(D)\right\}\frac{1}{f(D)}V$$

Rearranging the terms

$$xV = f(D)\left[x\frac{1}{f(D)}V\right] - \left\{\frac{d}{dD}f(D)\right\}\frac{1}{f(D)}V = f(D)\left[x\frac{1}{f(D)}V\right] - f'(D)\frac{1}{f(D)}V \quad (10)$$

Operating with $\frac{1}{f(D)}$ on both sides of (10), we get

$$\frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \frac{1}{f(D)}f'(D)\frac{1}{f(D)}V \quad (11)$$

Rewriting

$$\frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V \quad (12)$$

**General integration method**

Let us consider $f(D) \equiv (D - m_1)(D - m_2) \dots (D - m_n)$

Case(i)

The following formula may be established to find a $P.I$

$$P.I = e^{m_1 x}\int e^{(m_2 - m_1)x}\int e^{(m_3 - m_2)x}\int \dots \int e^{(m_n - m_{n-1})x}\int e^{-m_n x}\phi(x)(dx)^n \quad (13)$$

Case(ii)

This consists of expressing $\frac{1}{f(D)}$ as the sum of $n$ partial fractions as

$\frac{N_1}{D - m_1} + \frac{N_2}{D - m_2} + \dots + \frac{N_n}{D - m_n}$ then

$$P.I = N_1 e^{m_1 x}\int e^{-m_1 x}\phi(x)dx + N_2 e^{m_2 x}\int e^{-m_2 x}\phi(x)dx + \dots + N_n e^{m_n x}\int e^{-m_n x}\phi(x)dx \quad (14)$$

The above formulas are mentioned in [1].

**Method of Undetermined Coefficients**

The strategy of undetermined coefficients is to guess a trail solution for a particular integral, with coefficients to be decided, agreeing to the form of the right-hand side of the differential condition. The coefficients are then computed by substituting the trail solution into the differential equation.

**Method of Variation of Parameters**

Method of variation of parameters is the general method to find the general solution of the nonhomogeneous differential equation (with variable coefficients also) provided its complementary function is known.

Let us consider the complementary function $y_c$ is known as $y_c = c_1 y_1(x) + c_2 y_2(x)$, then $P.I$ is can be obtained as $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$ (15)

where

$u_1(x) = \int \frac{-\phi(x)y_2(x)}{W(x)}dx$, $u_2(x) = \int \frac{\phi(x)y_1(x)}{W(x)}dx$ and $W(x)$ =Wronskian of $y_1, y_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

The methodology of method undetermined coefficients and variation of parameters method are available in ([1]-[6])

The present paper is focused on the said methods in computing a $P.I$ of a differential equation and the comparison among the methods in view of first time learners. The formula (12) given by [4] applied to various equations to find out a $P.I$ if $\phi(x) = xV(x)$ where $V(x) = sinx$ or $cosx$ but its create a confusion while applying to some of the problems and the examples have shown in the following section. [6] has mentioned that P.I is not unique and the difference between any two $P.I$s of the same problem is a solution of the complementary function.

Due to misleading in applying the operators there exists more than one $P.I$s whereas the difference between any of the two will not be a solution of the complementary function, but the first time learners are not competent to decide that whether they obtained the correct $P.I$? or not? In view of these situations a general method has been suggested to compute the $P.I$ of type $x^k V(x)$ where $V(x) = \sin(ax + b)$ or $\cos(ax + b)$.

**2. Examples**

This section deals with some tests on a few examples that has been considered in several ODE textbooks. Here a $P.I$ has been determined using method of operators, general integration method, method of undermined coefficients and variation of parameters method.

First time learners may not interpret operations between the operators objectively and due to this, they may obtain different results. Some of the possible operations between the operators have been discussed with the help of a specific example and the results have compared with a desired$P.I$. Apart from the method of operators, general integration method, method of undetermined coefficients and method of variation of parameters have also been applied to obtain a$P.I$and compared the methods in view of the first time learners.

Example-1: Consider $y'' + y = xcosx$

The general solution of the equation is $y = c_1cosx + c_2sinx + \frac{xcosx}{4} + \frac{x^2sinx}{4} - \frac{1}{8}sinx$ and is taken from [4].

The complementary function is given as $y_c = c_1cosx + c_2sinx$ and a particular integral is given as $y_p = \frac{x^2}{4}sinx + \frac{x}{4}cosx - \frac{1}{8}sinx$.

The above differential equation is solved by applying method of operators, particularly applied the formula (12) and listed some of the possible results as they may be obtained by the first time learners due to unawareness of the applicability of operators.

Method-$S_1$:

$$\frac{1}{D^2+1}xcosx = x\frac{1}{D^2+1}cosx - \frac{2D}{\{(D^2+1)\}^2}cosx = xx\frac{1}{2D}cosx - x\frac{2D}{2(D^2+1)2D}cosx$$

$$= \frac{x^2}{2}sinx - x\frac{1}{2(D^2+1)}cosx = \frac{x^2}{2}sinx - xx\frac{1}{4D}cosx = \frac{x^2}{2}sinx - \frac{x^2}{4}sinx = \frac{x^2}{4}sinx$$

Method-$S_2$:

Let $y_p = \frac{1}{D^2+1}xcosx = x\frac{1}{D^2+1}cosx - \frac{2D}{\{(D^2+1)\}^2}cosx = xx\frac{1}{2D}cosx - \frac{2D}{2(D^2+1)2D}xcosx$

$$= \frac{x^2}{2}sinx - \frac{1}{2(D^2+1)}xcosx$$

$y_p + \frac{1}{2}y_p = \frac{x^2}{2}sinx \Rightarrow \frac{3}{2}y_p = \frac{x^2}{2}sinx \Rightarrow y_p = \frac{x^2}{3}sinx$

Method-$S_3$:

$$\frac{1}{D^2+1}xcosx = \frac{x^2}{2}sinx - \frac{2}{\{(D^2+1)\}^2}sinx$$

$$= \frac{x^2}{2} sinx - 2x \frac{1}{4D(D^2+1)} sinx = \frac{x^2}{2} sinx - \frac{x^2}{2} \frac{1}{(D^2+1)+2D^2} sinx$$
$$= \frac{x^2}{2} - \frac{x^2}{2} \frac{1}{-2} sinx = \frac{3}{4} x^2 sinx$$

Method-$S_4$:

$$\frac{1}{D^2+1} xcosx = x\frac{1}{D^2+1} cosx - \frac{2D}{\{(D^2+1)\}^2} cosx = \frac{x^2}{2} sinx - 2D\left[\frac{1}{(D^2+1)^2} cosx\right]$$
$$= \frac{x^2}{2} sinx - 2D\left[x\frac{1}{2(D^2+1)2D} cosx\right] = \frac{x^2}{2} sinx - \frac{1}{2}\left[\frac{1}{D(D^2+1)} cosx + x\frac{D}{D(D^2+1)} cosx\right]$$
$$= \frac{x^2}{2} sinx - \frac{1}{2}\left[\frac{1}{D^2+1} sinx + \frac{1}{D^2+1} cosx\right] = \frac{x^2}{2} sinx - \frac{1}{2}\left[x\frac{1}{2D} sinx + x^2\frac{1}{2D} cosx\right]$$
$$= \frac{x^2}{2} sinx - \frac{1}{4}[-xcosx + x^2 sinx] = \frac{x^2}{4} sinx + \frac{x}{4} cosx$$

Now applied the formula (11) and obtained the following result.

Method-$S_5$:

Let $y_p = \frac{1}{(D^2+1)} xcosx = x\frac{1}{D^2+1} cosx - \frac{1}{D^2+1}\ 2D\frac{1}{D^2+1} cosx$

$$= x\frac{1}{D^2+1} cosx - \left[\frac{1}{D^2+1}\left\{2D\left(\frac{1}{D^2+1} cosx\right)\right\}\right]$$
$$= \frac{x^2}{2} sinx - \left[\frac{1}{D^2+1}\left\{2D\left(x\frac{1}{2D} cosx\right)\right\}\right] = \frac{x^2}{2} sinx - \left[\frac{1}{D^2+1}\{D(xsinx)\}\right]$$
$$= \frac{x^2}{2} sinx - \frac{1}{D^2+1}\{sinx + xcosx\} = \frac{x^2}{2} sinx - \frac{1}{D^2+1} sinx - y_p \Rightarrow 2y_p$$
$$= \frac{x^2}{2} sinx - x\frac{1}{2D} sinx$$
$$= \frac{x^2}{2} sinx + \frac{x}{2} cosx \Rightarrow y_p = \frac{x^2}{4} sinx + \frac{x}{4} cosx$$

In the following methods $cosx$ has been replaced as $\frac{e^{ix}+e^{-ix}}{2}$ or real part of $e^{ix}$ and applied the formulas (4) and (7).

Method-$PS_1$:

$$\frac{1}{D^2+1} xcos = \frac{1}{D^2+1} xRe(e^{ix}) = Re.e^{ix}\frac{1}{(D+i)^2+1} x = Re.e^{ix}\frac{1}{D^2+2iD} x$$
$$= Re.e^{ix}\frac{1}{2iD}\left(1+\frac{D}{2i}\right)^{-1} x = Re.e^{ix}\frac{1}{2iD}\left(1-\frac{D}{2i}\right) x = Re.e^{ix}\frac{1}{2iD}\left(x-\frac{1}{2i}\right)$$
$$= Re.e^{ix}\frac{1}{2i}\left(\frac{x^2}{2}-\frac{x}{2i}\right)$$
$$= \frac{x^2}{4} sinx + \frac{x}{4} cosx$$

Method-$PS_2$:

$$\frac{1}{D^2+1} xcos = Re.\frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right] xe^{ix}$$
$$= Re.e^{ix}\frac{1}{2i}\left[\frac{1}{D} - \frac{1}{D+2i}\right] x = Re.e^{ix}\frac{1}{2i}\left[\frac{x^2}{2} - \frac{1}{2i}\left(1+\frac{D}{2i}\right)^{-1} x\right]$$
$$= Re.e^{ix}\frac{1}{2i}\left[\frac{x^2}{2} - \frac{1}{2i}\left(x-\frac{1}{2i}\right)\right] = \left(-\frac{i}{2} cosx + \frac{1}{2} sinx\right)\left[\left(\frac{x^2}{2} - \frac{1}{4}\right) - \frac{1}{i}\frac{x}{2}\right]$$

$$= \frac{x^2}{4} sinx + \frac{x}{4} cosx - \frac{sinx}{8}$$

Method-$PS_3$:

$$\frac{1}{D^2+1} xcos = \frac{1}{D^2+1} x \frac{e^{ix} + e^{-ix}}{2} = \frac{1}{2}\left[\frac{1}{D^2+1} xe^{ix} + \frac{1}{D^2+1} xe^{-ix}\right]$$

Using $PS_2$ it is obtained as

$$= \frac{1}{2}\left[\frac{e^{ix}}{2i}\left(\frac{x^2}{2} - \frac{x}{2i}\right) - \frac{1}{2i} e^{-ix}\left(\frac{x^2}{2} + \frac{x}{2i}\right)\right]$$

$$= \frac{x^2}{4} sinx + \frac{x}{4} cosx$$

General integration method has been applied using (13) and (14) in the following methods.

Method-$G_1$:

$$\frac{1}{D^2+1} xcos = Re. \frac{1}{D^2+1} xe^{ix} = Re. \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right] xe^{ix}$$

$$= Re. \frac{1}{2i}\left[e^{ix}\int xe^{ix}e^{-ix}dx - e^{-ix}\int xe^{ix}e^{ix}dx\right]$$

$$= \frac{x^2}{4} sinx + \frac{x}{4} cosx - \frac{sinx}{8}$$

Method-$G_2$:

$$\frac{1}{D^2+1} xcos = \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right] xcosx$$

$$= \frac{1}{2i}\left[e^{ix}\int xcosxe^{-ix}dx - e^{-ix}\int xcosxe^{ix}dx\right]$$

Evaluation of integrals consisting of three functions is a difficult task but one can obtain

$$= \frac{1}{2i}\left[e^{ix}\left(\frac{e^{-2ix}}{8} + \frac{ix}{4}e^{-2ix} + \frac{x^2}{4}\right) - e^{-ix}\left(\frac{e^{2ix}}{8} - \frac{ix}{4}e^{2ix} + \frac{x^2}{4}\right)\right]$$

$$= \frac{x^2}{4} sinx + \frac{x}{4} cosx - \frac{sinx}{8}$$

Method-$G_3$:

$$\frac{1}{D^2+1} xcosx = Re. \frac{1}{(D+i)(D-i)} xe^{ix}$$

$$= Re. e^{-ix}\int e^{ix}\left(e^{ix}\int xe^{ix}e^{-ix}dx)\right)dx$$

$$= Re. \frac{-i}{16} e^{ix}(4x^2 + 4ix - 2) = \frac{x^2}{4} sinx + \frac{x}{4} cosx - \frac{sinx}{8}$$

Method-$G_4$:

$$\frac{1}{D^2+1} xcosx = \frac{1}{(D+i)(D-i)} xcosx$$

$$= e^{-ix}\int e^{ix}\left(e^{ix}\int xcosxe^{-ix}dx)\right)dx$$

On evaluating the integral it is obtained as

$$= \frac{x}{8}\left(e^{ix} + e^{-ix}\right) - \frac{ix^2}{8}\left(e^{ix} - e^{-ix}\right) + \frac{i}{16} e^{ix}$$

$$=\frac{x^2}{4}sinx+\frac{x}{4}cosx+\frac{i}{16}cosx-\frac{1}{16}sinx$$

Method-$G_5$:

$$\frac{1}{D^2+1}xcosx=\frac{1}{2i}\left[\frac{1}{(D-i)}-\frac{1}{(D+i)}\right]x\frac{e^{ix}+e^{-ix}}{2}$$
$$=\frac{1}{2i}\left[e^{ix}\int x\frac{e^{ix}+e^{-ix}}{2}e^{-ix}dx-e^{-ix}\int x\frac{e^{ix}+e^{-ix}}{2}e^{ix}dx\right]$$
$$=\frac{1}{2i}\left[\frac{x^2}{4}e^{ix}+\frac{e^{-ix}}{8}+\frac{ix}{4}e^{-ix}-\left(\frac{x^2}{4}e^{-ix}+\frac{e^{ix}}{8}-\frac{ix}{4}e^{ix}\right)\right]$$
$$=\frac{x^2}{4}sinx+\frac{x}{4}cosx-\frac{sinx}{8}$$

Method-$G_6$:

$$\frac{1}{D^2+1}xcosx=\frac{1}{(D+i)(D-i)}x\frac{e^{ix}+e^{-ix}}{2}$$
$$=e^{ix}\int\left[e^{-ix}\left\{e^{-ix}\int e^{ix}\frac{x(e^{ix}+e^{-ix})}{2}dx\right\}\right]dx=e^{ix}\int\left[e^{-ix}\left\{e^{-ix}\int x\frac{(e^{2ix}+1)}{2}dx\right\}\right]dx$$
$$=\frac{e^{ix}}{2}\int\left\{e^{-2ix}\left(\frac{e^{2ix}}{4}-\frac{ixe^{2ix}}{2}+\frac{x^2}{2}\right)\right\}dx=\frac{e^{ix}}{2}\int\left(\frac{1}{4}+\frac{x^2e^{-2ix}}{2}-\frac{ix}{2}\right)dx$$
$$=\frac{e^{ix}}{2}\left(\frac{x}{4}-\frac{ie^{-2ix}}{8}+\frac{xe^{-2ix}}{4}+\frac{ix^2e^{-2ix}}{4}-\frac{ix^2}{4}\right)$$
$$=\frac{x}{4}\left(\frac{e^{ix}+e^{-ix}}{2}\right)+\frac{x^2}{4}(-i)\left(\frac{e^{ix}-e^{-ix}}{2}\right)-\frac{i}{16}e^{-ix}$$
$$=\frac{x}{4}cosx+\frac{x^2}{4}sinx-\frac{i}{16}(cox-isinx)=\frac{x^2}{4}sinx+\frac{x}{4}cosx-\frac{i}{16}cosx-\frac{1}{16}sinx$$

To apply method of undetermined coefficients one can assume a $P.I$ is in the form as $y=Ax^2sinx+Bx^2cosx+Cxsinx+Dxcosx$ and obtained a $P.I$ as follows

Method-$U_1$:

$Esinx+Fcosx$is not included in the assumed $P.I$since these terms are in the complementary function. On substituting $y$ in the differential equation one can obtain

$$(2A-2D)sinx+(2B+2C)cosx+4Axcosx-4Bxsinx=xcosx$$

$2A-2D=0;\ 2B+2C=0; 4A=1$ and $4B=0$ on solving these equations

$A=\frac{1}{4}=D$ and $B=0=C$

Hence $P.I=\frac{x^2}{4}sinx+\frac{x}{4}cosx$

On applying the variation of parameters a $P.I$ is obtained as follows

Method-$V_1$:

$C.F=c_1cosx+c_2sinx$ let $y_1=cosx$ and $y_2=sinx$ then $W=1$

$$u_1=-\int sinxxcosxdx=\frac{x}{4}cos2x-\frac{1}{8}sin2x$$
$$u_2=\int cosxxcosxdx=\frac{x^2}{4}+\frac{x}{2}sin2x+\frac{1}{4}cos2x$$
$$P.I=u_1y_1+u_2y_2=\frac{x^2}{4}sinx+\frac{x}{4}cosx-\frac{sinx}{4}+\frac{sinx\cos^2x}{4}$$

This $P.I$ may not be considered as it is not satisfied the considered equation.

A desired particular integral is obtained by applying method of operators using formula (11), $PS_1 - PS_3$, general integration method $(G1 - G_6)$ and method of undetermined coefficients $(U_1)$. The results obtained in the said methods may differ with $k_1 cosx + k_2 sinx$ where $k_1, k_2$ are constants and it is the solution of the complementary equation. However, the result obtained using variation of parameters may not satisfy the differential equation hence it may not be a desired particular integral.

If $\phi(x) = x^k\{\cos(ax+b) \text{ or } \sin(ax+b)\}$ for $k \in \mathbb{N}\backslash\{1\}$ then theorem-5 can be extended as follows:

Theorem: If $\phi(x) = x^k V(x)$ then

$$\frac{1}{f(D)}x^k V = x^k \frac{1}{f(D)}V - k_{c_1}\frac{1}{f(D)}\left(x^{k-1}f'(D)\frac{1}{f(D)}V\right) - k_{c_2}\frac{1}{f(D)}\left(x^{k-2}f''(D)\frac{1}{f(D)}V\right) - \cdots$$
$$-k_{c_{k-1}}\frac{1}{f(D)}\left(x^1 f^{(k-1)}(D)\frac{1}{f(D)}V\right) - k_{c_k}\frac{1}{f(D)}\left(f^{(k)}(D)\frac{1}{f(D)}V\right). \qquad (16)$$

Proof: in the similar lines of proof of theorem-5 it can be obtained

$$f(D)\left(x^k U\right) = k_{c_0}x^k f(D)U + k_{c_1}x^{k-1}\left\{\frac{d}{dD}f(D)\right\}U + k_{c_2}x^{k-2}\left\{\frac{d^2}{dD^2}f(D)\right\}U + \cdots$$
$$+ k_{c_{k-1}}x\left\{\frac{d^{(k-1)}}{dD^{(k-1)}}f(D)\right\}U + k_{c_k}\left\{\frac{d^k}{dD^k}f(D)\right\}U$$

Put $U = \frac{1}{f(D)}V$ and on rearranging the terms it is obtained as

$$\frac{1}{f(D)}x^k V = x^k\frac{1}{f(D)}V - k_{c_1}\frac{1}{f(D)}\left[x^{k-1}\left\{\frac{d}{dD}f(D)\right\}\frac{1}{f(D)}V\right]$$
$$- k_{c_2}\frac{1}{f(D)}\left[x^{k-2}\left\{\frac{d^2}{dD^2}f(D)\right\}\frac{1}{f(D)}V\right] - \cdots$$
$$- k_{c_{k-1}}\frac{1}{f(D)}\left[x\left\{\frac{d^{(k-1)}}{dD^{(k-1)}}f(D)\right\}\frac{1}{f(D)}V\right] - k_{c_k}\frac{1}{f(D)}\left[\left\{\frac{d^k}{dD^k}f(D)\right\}\frac{1}{f(D)}V\right]$$

Example-2: $y'' - 4y' + 4y = 8e^{2x}x^2 sin2x$
Solution: CF= $(c_1 + c_2 x)e^{2x}$

Using (13) or (4) one can obtain

$$P.I = \frac{1}{D^2 - 4D + 4}8e^{2x}x^2 sin2x = 8e^{2x}\int\left[\int x^2 sin2x dx\right]dx$$
$$= -e^{2x}[(2x^2 - 3)sin2x + 4xcos2x]$$

To obtain a particular integral using the above method one need to do computations using integration and it may become cumbersome.

Using (4) and (16) first time learners may write

$$P.I = \frac{1}{D^2 - 4D + 4}8e^{2x}x^2 sin2x = 8e^{2x}\frac{1}{D^2}x^2 sin2x$$
$$= 8e^{2x}\left[x^2\frac{1}{D^2}sin2x - 2x\frac{2D}{(D^2)^2}sin2x - \frac{2}{(D^2)^2}sin2x\right]$$

Using (5) it may obtained$P.I = e^{2x}[-sin2x(2x^2 - 1) - 4xcos2x]$, but it will not be a desired particular integral as if it is substituted in the given equation one can obtain $8e^{2x}(x^2 + 1)sin2x$.

Therefore, first time learners need to understand the formula (16) properly before applying it to find a particular integral and need to written as follows:

$$P.I = \frac{1}{f(D)}x^2 sin2x = x^2\frac{1}{f(D)} - 2\frac{1}{f(D)}\left[xf^{'}(D)\left\{\frac{1}{f(D)}sin2x\right\}\right] - \frac{1}{f(D)}\left[f^{''}(D)\left\{\frac{1}{f(D)}sin2x\right\}\right]$$

$$= x^2\frac{1}{D^2}sin2x - 2\frac{1}{D^2}\left[x\, 2D\frac{1}{D^2}sin2x\right] - \frac{1}{D^2}\left[2\frac{1}{D^2}sin2x\right]$$

$$= -\frac{x^2}{4}sin2x + 2\frac{1}{D^2}(xcos2x) - \frac{1}{8}sin2x$$

Here it is required to use (11) instead of (12) to obtain correct particular integral as

$$PI = -e^{2x}[(2x^2-3)sin2x + 4xcos2x]$$

To avoid the confusion in applying (16) and cumbersome integration as mentioned above it is suggested to write as follows:

$$P.I = \frac{1}{D^2-4D+4}8e^{2x}x^2 sin2x = Im.8\frac{1}{D^2-4D+4}e^{(2+2i)x}x^2$$

Now apply (4) and (7) to obtain $P.I = -e^{2x}[(2x^2-3)sin2x + 4xcos2x]$

Example-3: $y^{''} + y = x^k cosx$ where $k \in \mathbb{N}\backslash\{1\}$

Example-4: $y^{iv} + 2y^{''} + y = xcosx$

If $k \in \mathbb{N}\backslash\{1\}$ then applying method of operators using formula (16) to obtain a desired particular integral for $\phi(x) = x^k\{\cos(ax+b)$ or $\sin(ax+b)\}$ is not an easy task for the first time learners while applying the methods discussed in $PS_2 - PS_3$, $G_1 - G_6$, method of undetermined coefficients and variation of parameters have also become tedious.

## 3. Conclusion

Of the cited methods 12, 13, 14 and 15 above, when no explicit instructions on a particular method were given while solving the differential equation, the outcomes are expected to be matching with example, $S_1, S_2, S_3$ and $V_1$. These examples may lead to a bit of frustration when time bound examinations take place for competitive scoring under mandatory University examinations and subsequent grading against a compulsory course like, Engineering Mathematics. This article is an attempt to moderate continuous attempts under numerous factual methods by the student from respective learnings based on huge mathematics literature and prescribed University textbooks, and at the same time, ease the conundrum of methods for the first time learners. In this context, the method $PS_1$ could be of rational help in order to obtain particular integral of the type $\phi(x) = x^k\{\cos(ax+b)$ or $\sin(ax+b)\}$. It is believed that a focused attempt using the method $PS_1$ should be of more academic help the student while obtaining particular integral of the above examples for higher order differential equations.

### References


[1] Ayres, J. R., "Theory and Problems of Differential Equations", McGraw-Hill, 1952.
[2] Coddington, E.A., "An Introduction to Ordinary Differential Equations", Prentice Hall of India, ISBN 812030361, 2005.
[3] Kreyszig, E., "Advanced Engineering Mathematics (10th ed.)", John Wiley & Sons, ISBN 9780470458365, 2011.
[4] Ramana, B. V., "Higher Engineering Mathematics (6th ed.)", Tata McGraw-Hill, ISBN 9780071070089, 2014.
[5] Raisinghania, M.D., "Ordinary and Partial Differential Equations(19th ed.)", S. Chand, ISBN 9789352535866, 2017.
[6] Xie, W. C., "Differential Equations for Engineers (1st ed.)", Cambridge University Press, ISBN ,2010.